Three-Body Regularization

Initial conditions $\mathbf{r}_i,\ \mathbf{p}_i, \quad \mathbf{p}_3 \ = \ -(\mathbf{p}_1 + \mathbf{p}_2)$

Basic Hamiltonian $\mu_{k3} \ = \ \dfrac{m_k m_3}{m_k + m_3}$

$$H \ = \ \sum_{k=1}^{2} \frac{1}{2\mu_{k3}} \mathbf{p}_k^2 + \frac{1}{m_3} \mathbf{p}_1 \cdot \mathbf{p}_2 - \frac{m_1 m_3}{R_1} - \frac{m_2 m_3}{R_2} - \frac{m_1 m_2}{R}$$

KS coordinate transformation $\mathbf{Q}_k^2 \ = \ R_k \,, \quad (k = 1, 2)$

Time transformation $dt \ = \ R_1 R_2 \, d\tau$

Regularized Hamiltonian $\Gamma^* \ = \ R_1 R_2 \, (H - E_0)$

$$\Gamma^* \ = \ \sum_{k=1}^{2} \frac{1}{8\mu_{k3}} R_l \, \mathbf{P}_k^2 \ + \ \frac{1}{16 m_3} \mathbf{P}_1^T \mathbf{A}_1 \cdot \mathbf{A}_2 \mathbf{P}_2$$

$$- \, m_1 m_3 R_2 - m_2 m_3 R_1 - \frac{m_1 m_2 R_1 R_2}{|\mathbf{R}_1 - \mathbf{R}_2|} - E_0 R_1 R_2$$

Equations of motion

$$\frac{d\mathbf{Q}_k}{d\tau} \ = \ \frac{\partial \Gamma^*}{\partial \mathbf{P}_k} \,, \qquad \frac{d\mathbf{P}_k}{d\tau} \ = \ - \frac{\partial \Gamma^*}{\partial \mathbf{Q}_k}$$

Regular solutions: $R_1 \to 0$ or $R_2 \to 0$

Singular term < regular terms: $|\mathbf{R}_1 - \mathbf{R}_2| > \ \max\,(R_1,\ R_2)$

Three-Body Transformations

Coordinates & momenta $\mathbf{q}_k = \tilde{\mathbf{q}}_k - \tilde{\mathbf{q}}_3, \quad \mathbf{p}_k = \tilde{\mathbf{p}}_k$

Regularized coordinates ($q_1 \geq 0$)

$$\begin{aligned} Q_1 &= [\tfrac{1}{2}(|\mathbf{q}_1| + q_1)]^{1/2} \\ Q_2 &= \tfrac{1}{2} q_2 / Q_1 \\ Q_3 &= \tfrac{1}{2} q_3 / Q_1 \\ Q_4 &= 0 \end{aligned}$$

Regularized momenta $\mathbf{P}_k = \mathbf{A}_k \, \mathbf{p}_k$

Basic matrix

$$\mathbf{A}_1 = 2 \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ -Q_2 & Q_1 & Q_4 & -Q_3 \\ -Q_3 & -Q_4 & Q_1 & Q_2 \\ Q_4 & -Q_3 & Q_2 & -Q_1 \end{bmatrix}$$

KS transformations $\mathbf{q}_k = \frac{1}{2} \mathbf{A}_k^{\mathrm{T}} \mathbf{Q}_k$

Physical momenta $\mathbf{p}_k = \frac{1}{4} \mathbf{A}_k^{\mathrm{T}} \mathbf{P}_k / R_k$

Coordinates & momenta

$$\begin{aligned} \tilde{\mathbf{q}}_3 &= -\sum_{k=1}^{2} m_k \mathbf{q}_k / M \\ \tilde{\mathbf{q}}_k &= \tilde{\mathbf{q}}_3 + \mathbf{q}_k \\ \tilde{\mathbf{p}}_k &= \mathbf{p}_k \\ \tilde{\mathbf{p}}_3 &= -(\mathbf{p}_1 + \mathbf{p}_2) \qquad (k = 1, 2) \end{aligned}$$

Perturbed Three-Body Regularization

Regularized Hamiltonian

$$\Gamma^* = R_1 R_2 (H_3 + \mathcal{R} - E)\,, \quad E_3 = E - \mathcal{R}$$

New equations of motion

$$\frac{d\mathbf{Q}_k}{d\tau} = \frac{\partial (R_1 R_2 H_3)}{\partial \mathbf{P}_k}$$

$$\frac{d\mathbf{P}_k}{d\tau} = -(H_3 - E_3)\frac{\partial (R_1 R_2)}{\partial \mathbf{Q}_k} - R_1 R_2 \frac{\partial}{\partial \mathbf{Q}_k}(H_3 + \mathcal{R})$$

External perturbation for Plummer model

$$\frac{\partial \mathcal{R}}{\partial \mathbf{Q}_k} = \sum_{i=1}^{3} \frac{\partial \mathcal{R}}{\partial \mathbf{r}_i}\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_k}\frac{\partial \mathbf{q}_k}{\partial \mathbf{Q}_k}\,, \qquad \frac{\partial \mathcal{R}}{\partial \mathbf{r}_i} = -\frac{m_i M_{\mathrm{p}} \mathbf{r}_i}{(r_i^2 + \epsilon^2)^{3/2}}$$

Transformations and c.m. condition $\mathbf{r}_{\mathrm{cm}} = \sum m_i \mathbf{r}_i / M$

$$\mathbf{r}_1 = \mathbf{r}_{\mathrm{cm}} + (m_2 + m_3)\mathbf{q}_1/M - m_2 \mathbf{q}_2/M$$

$$\mathbf{r}_2 = \mathbf{r}_{\mathrm{cm}} - m_1 \mathbf{q}_1/M + (m_1 + m_3)\mathbf{q}_2/M$$

$$\mathbf{r}_3 = \mathbf{r}_{\mathrm{cm}} - m_1 \mathbf{q}_1/M - m_2 \mathbf{q}_2/M$$

Application of $\partial \mathbf{r}_i / \partial \mathbf{q}_k$ yields mass ratios

Motion of c.m. $\quad \dfrac{d\mathbf{v}_{\mathrm{cm}}}{d\tau} = -R_1 R_2 M_{\mathrm{p}} \sum \dfrac{m_i \mathbf{r}_i}{(r_i^2 + \epsilon^2)^{3/2}}$

Basic transformation $\mathbf{q}_k = \mathbf{A}_k^{\mathrm{T}}\mathbf{Q}_k/2$ gives $\partial\mathbf{q}_k/\partial\mathbf{Q}_k = \mathbf{A}_k$

Combining terms

$$\frac{\partial\mathcal{R}}{\partial\mathbf{Q}_k} = -\frac{\mathbf{A}_k m_k}{M}[m_l(\mathbf{F}_k - \mathbf{F}_l) + m_3(\mathbf{F}_k - \mathbf{F}_3)]\,, \quad l = 3-k$$

Internal energy change

$$\frac{dE_3}{d\tau} = -\frac{d\mathcal{R}}{d\tau}$$

Conversion to known expressions

$$\frac{d\mathcal{R}}{d\tau} = \sum_{k=1}^{2}\frac{\partial\mathcal{R}}{\partial\mathbf{Q}_k}\frac{d\mathbf{Q}_k}{d\tau}$$

Substitution $\quad \dfrac{d\mathbf{Q}_k}{d\tau} = \dfrac{1}{4\mu_{k3}}R_l\mathbf{P}_k + \dfrac{1}{16m_3}\mathbf{A}_k\mathbf{A}_l^{\mathrm{T}}\mathbf{P}_l$

Orthogonality condition

$$\mathbf{A}_k\mathbf{A}_k^{\mathrm{T}} = 4R_k$$

Final energy derivative

$$\frac{d\mathcal{R}}{d\tau} = -\frac{1}{4}\sum_{k=1}^{2}R_l\mathbf{P}_k^{\mathrm{T}}\mathbf{A}_k(\mathbf{F}_k - \mathbf{F}_3)$$

Note $\partial\mathcal{R}/\partial\mathbf{Q}_k$ used for $\mathbf{P}_k^{'}$ and $E_3^{'}$

Consistency check: $\quad \Delta E = H_3 - E_3$

TRIPLE2 Features

Time reversal	Strict accuracy test: $\sigma_{\mathrm{x}},\ \sigma_{\mathrm{v}}$
`X11` movie	**`make xtriple`**
`PGPLOT` movie	**`make ptriple`**
External perturbation	Plummer model: $M_{\mathrm{p}},\ a_{\mathrm{p}}$
Closest encounter	Osculating two-body separation
Physical collision	Iteration for small R_k (project)
Physical units	Introduce $M^*,\ L^*, \quad \Rightarrow T^*,\ V^*$
Post-Newtonian terms	$c = \dfrac{3 \times 10^5}{V^*}$, (project)

Post-Newtonian Terms

Equation of motion $$\frac{d^2\mathbf{r}}{dt^2} = \frac{M}{r^2}\left[(-1+A)\frac{\mathbf{r}}{r} + B\mathbf{v}\right]$$

First-order precession $$M = m_1 + m_2, \quad \eta = \frac{m_1 m_2}{M^2}$$

$$A_1 = 2(2+\eta)\frac{M}{r} - (1+3\eta)v^2 + \frac{3}{2}\eta\dot{r}^2$$

$$B_1 = 2(2-\eta)\dot{r}$$

Second-order precession $A_2 = ..., \quad B_2 = ...$

Gravitational radiation

$$A_{5/2} = \frac{8}{5}\eta\frac{M}{r}\dot{r}\left(\frac{17M}{3r} + 3v^2\right)$$

$$B_{5/2} = -\frac{8}{5}\eta\frac{M}{r}\left(3\frac{M}{r} + v^2\right)$$

Total GR perturbation

$$\mathbf{P}_{GR} = \frac{m_1 m_2}{c^2 r^2}\left[(A_1 + \frac{A_2}{c^2} + \frac{A_{5/2}}{c^3})\frac{\mathbf{r}}{r} + (B_1 + \frac{B_2}{c^2} + \frac{B_{5/2}}{c^3})\mathbf{v}\right]$$

Radiation energy loss

$$\Delta E_{GR} = \int \mathbf{P}_{GR}\cdot\mathbf{v}\,dt$$

Time-scale for shrinkage $$\tau_{GR} = 1.3\times 10^{18}\frac{a^4}{m_1 m_2 M}\frac{(1-e^2)^{7/2}}{4.5}$$

Kozai cycles $A_2,\ B_2$ activated if $\tau_{\rm Kozai} < 0.01\,\tau_{GR}$

Alternative Transformations

Improper integral

$$\tau = \int \frac{dt}{R_1 R_2} \text{ for } R_1, R_2 \to 0$$

Modified time transformation

$$t' = \frac{R_1 R_2}{(R_1 + R_2)^{1/2}}; \quad t' \propto R^{3/2}$$

$$\dot{R} \propto R^{-1/2}, \quad R^{3/2} \propto t, \quad \Rightarrow \tau \propto ln\, t$$

Potential energy choice $\quad t' = \dfrac{1}{U}$

Explicit time relation $\quad t = -\dfrac{\tau + C}{2E} + \dfrac{1}{2E} \displaystyle\sum_{i=1}^{i=3} \mathbf{r}_i \cdot \mathbf{p}_i$

Regular expression $\quad t = \left(\dfrac{1}{2} \displaystyle\sum_{k=1}^{k=2} \mathbf{Q}_k \mathbf{P}_k - C - \tau \right) /2E$

Lagrangian choice $\quad t' = \dfrac{1}{T - U}$