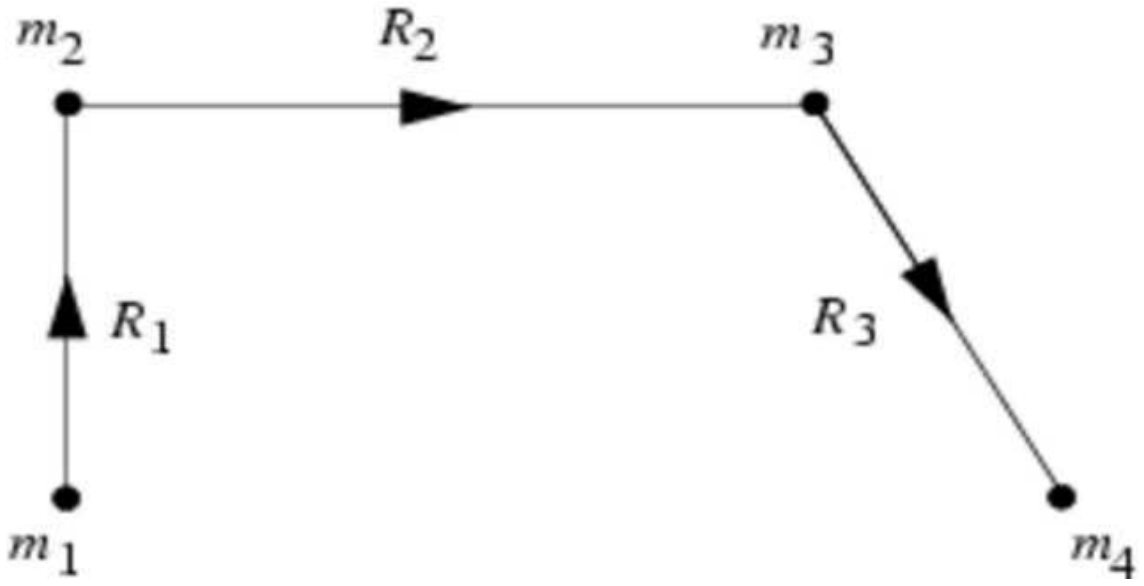
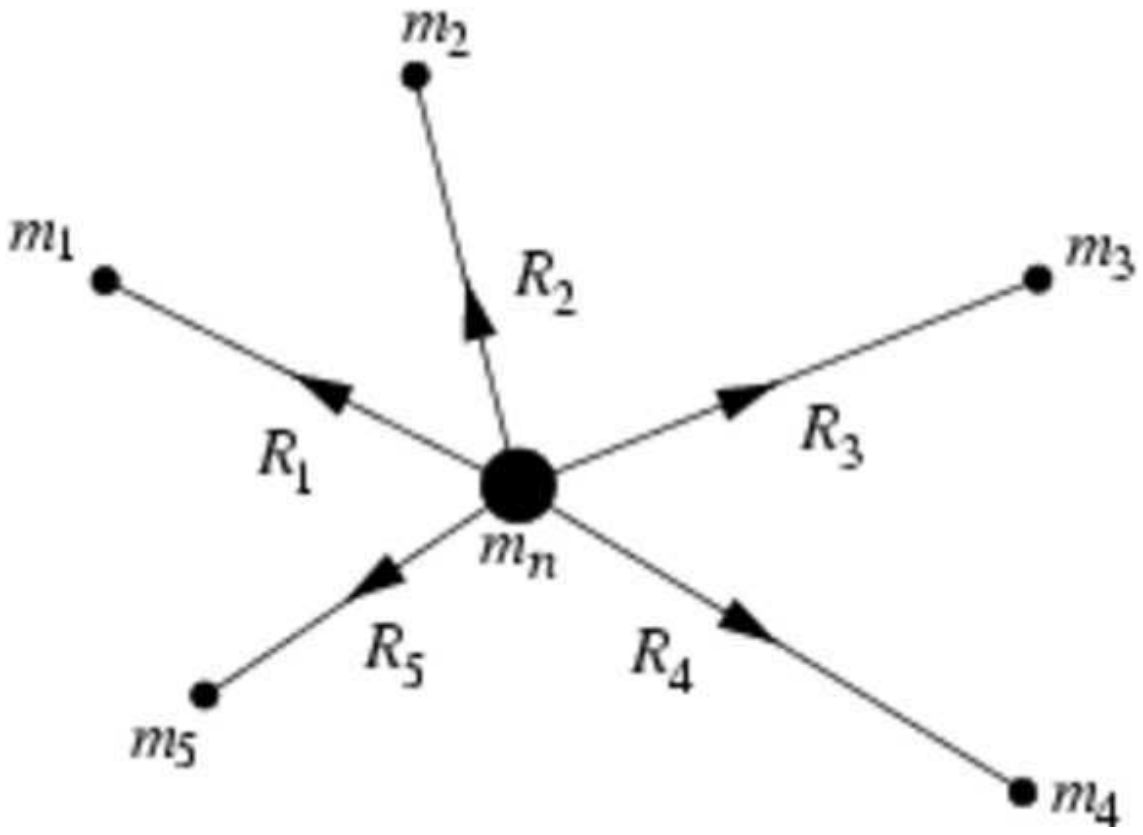

m_2
R_2
m_3
R_1
R_3
m_1
m_4


m_2
m_1
R_2
m_3
R_1
R_3
m_n
R_5
R_4
m_5
m_4

Wheel-Spoke Regularization

Initial conditions $\quad m_i,\ \tilde{\mathbf{q}}_i,\ \tilde{\mathbf{p}}_i, \quad i = 0, \ldots, N, \quad n \le N$

Generating function $\quad W(\mathbf{q}_i, \tilde{\mathbf{p}}_i) = \sum_{i=1}^{N} \tilde{\mathbf{p}}_i \cdot \mathbf{q}_i + \left(\sum_{i=0}^{N} \tilde{\mathbf{p}}_i \right) \cdot \mathbf{q}_0$

Hamiltonian $\quad H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2\mu_i} + \frac{1}{m_0} \sum_{i<j}^{N} \mathbf{p}_i^{\mathrm{T}} \cdot \mathbf{p}_j - m_0 \sum_{i=1}^{N} \frac{m_i}{R_i} - \sum_{i<j}^{N} \frac{m_i m_j}{R_{ij}}$

Canonical variables $\quad W(\mathbf{p}_i, \mathbf{Q}_i) = \sum_{i=1}^{N} \mathbf{p}_i^{\mathrm{T}} \cdot \mathbf{f}_i(\mathbf{Q}_i)$

Regularized momenta $\quad \mathbf{P}_i = \mathbf{A}_i \mathbf{p}_i\,, \quad (i = 1, \ldots, n)$

Coordinates and momenta $\quad \mathbf{q}_i = \tilde{\mathbf{q}}_i - \tilde{\mathbf{q}}_0, \quad \mathbf{p}_i = \tilde{\mathbf{p}}_i$

Inverse transformations $\quad \mathbf{q}_i = \frac{1}{2}\,\mathbf{A}_i^{\mathrm{T}} \mathbf{Q}_i, \quad \mathbf{p}_i = \frac{1}{4}\,\mathbf{A}_i^{\mathrm{T}} \mathbf{P}_i / R_i$

Local coordinates & momenta

$$
\begin{aligned}
\tilde{\mathbf{q}}_0 &= -\sum_{i=1}^{n} m_i \mathbf{q}_i \Big/ \sum_{i=0}^{n} m_i \\
\tilde{\mathbf{q}}_i &= \tilde{\mathbf{q}}_0 + \mathbf{q}_i \\
\tilde{\mathbf{p}}_i &= \mathbf{p}_i\,, \qquad (i = 1, \ldots, n) \\
\tilde{\mathbf{p}}_0 &= -\sum_{i=1}^{n} \mathbf{p}_i
\end{aligned}
$$

Wheel-Spoke Implementation

Select members	$\Delta t_{\rm cm} < \Delta t_{\rm cl}, \quad R = a\,(1+e)$
Initialize in local c.m.	$\sum m_i\,\mathbf{r}_i, = 0, \quad \sum m_i\,\dot{\mathbf{r}}_i = 0$
Chain indices & vectors	$\mathbf{Q}$, $\mathbf{P}$, $\quad N_{\rm eq} = 8\,(N-1)$
Define useful quantities	$T_{\rm cr}$, $R_{\rm grav}$, $\Delta\tau_0$
Softening of singularities	$\epsilon = f\,R_{\rm grav}, \quad \Rightarrow\ E = {\rm const}$
Form perturber list	$d < \left(\dfrac{2m}{M_{\rm ch}\gamma_0}\right)^{1/3} R_{\rm grav}$
Check time-step	$\Delta\tau = \int L\,dt, \quad L = T - \Phi$
B-S integration step	$\mathbf{r}_i = ((\frac{1}{6}\dot{\mathbf{F}}_i\delta t_i + \frac{1}{2}\mathbf{F}_i)\delta t_i + \dot{\mathbf{r}}_i)\delta t_i$
Physical variables	$\mathbf{R}_k = \dfrac{1}{2}\,\mathbf{A}_k\,\mathbf{Q}_k, \quad \mathbf{p}_k = \dfrac{1}{4}\,\dfrac{\mathbf{A}_k\,\mathbf{P}_k}{\mathbf{Q}_k^2}$
Addition of member	$\gamma > 0.05, \quad r_p \le \sum R_k$
Termination test	$\dot{R}^2 > 2M/R, \quad R > R_{\rm cl}$
Continue N-body integration	$t > t_{\rm max} = t_{\rm blk}$

PN Decision-Making

Equation of motion $$\frac{d^2\mathbf{r}}{dt^2} = \frac{M}{r^2}\left[(-1+A)\frac{\mathbf{r}}{r} + B\mathbf{v}\right]$$

Classical form $$\mathbf{F} = \mathbf{F}_0 + \frac{\mathbf{F}_2}{c^2} + \frac{\mathbf{F}_4}{c^4} + \frac{\mathbf{F}_5}{c^5}$$

GR radiation time-scale $$t_{\rm GR} = \frac{5}{64}\frac{c^5\, g(e)\, a^4}{X(1+X)\, m_N^3}, \quad c = \frac{3\times 10^5}{V^*}$$

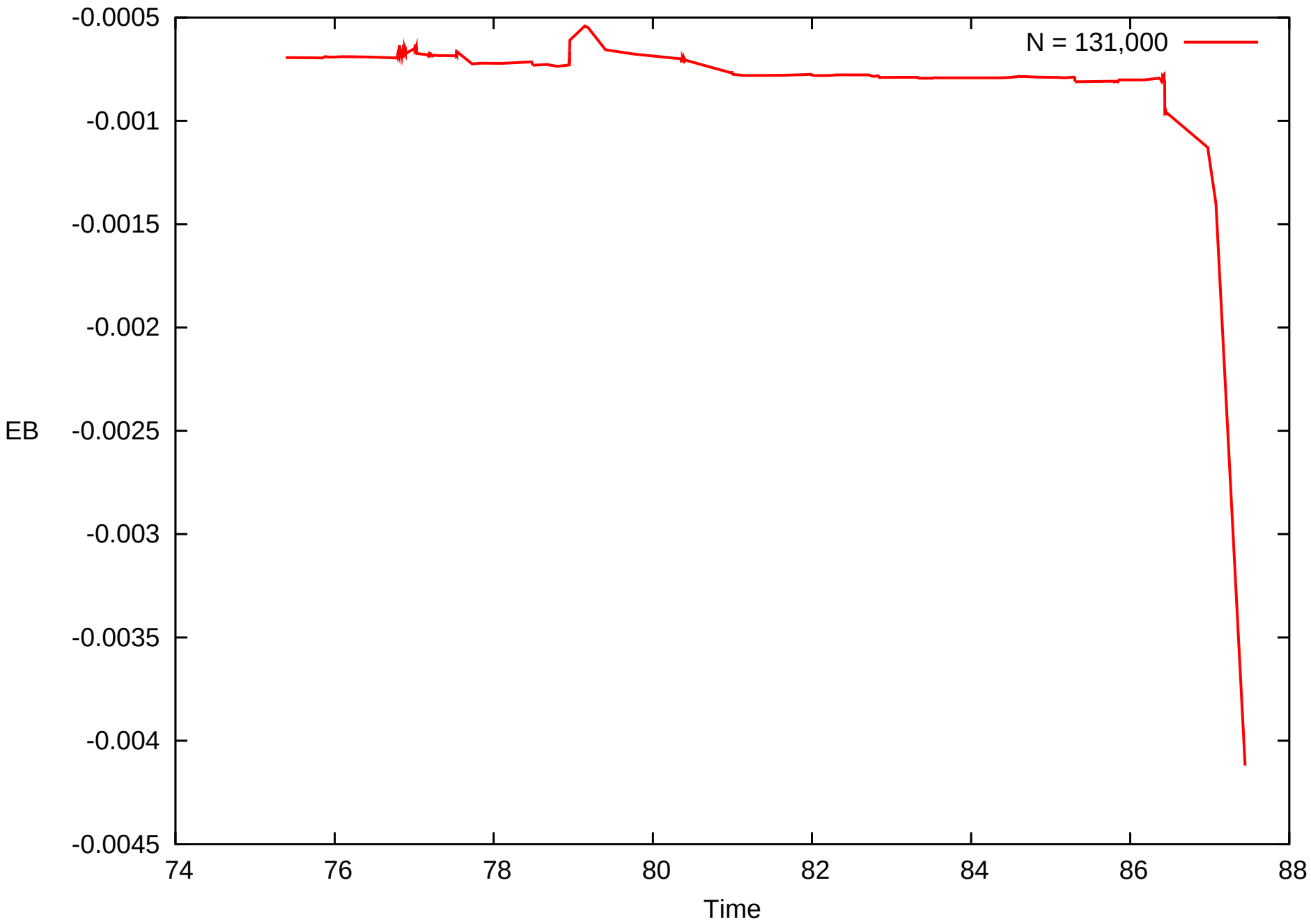
$$g(e) \simeq \frac{(1-e^2)^{7/2}}{4.35}, \quad X = \frac{m_i}{m_N}$$

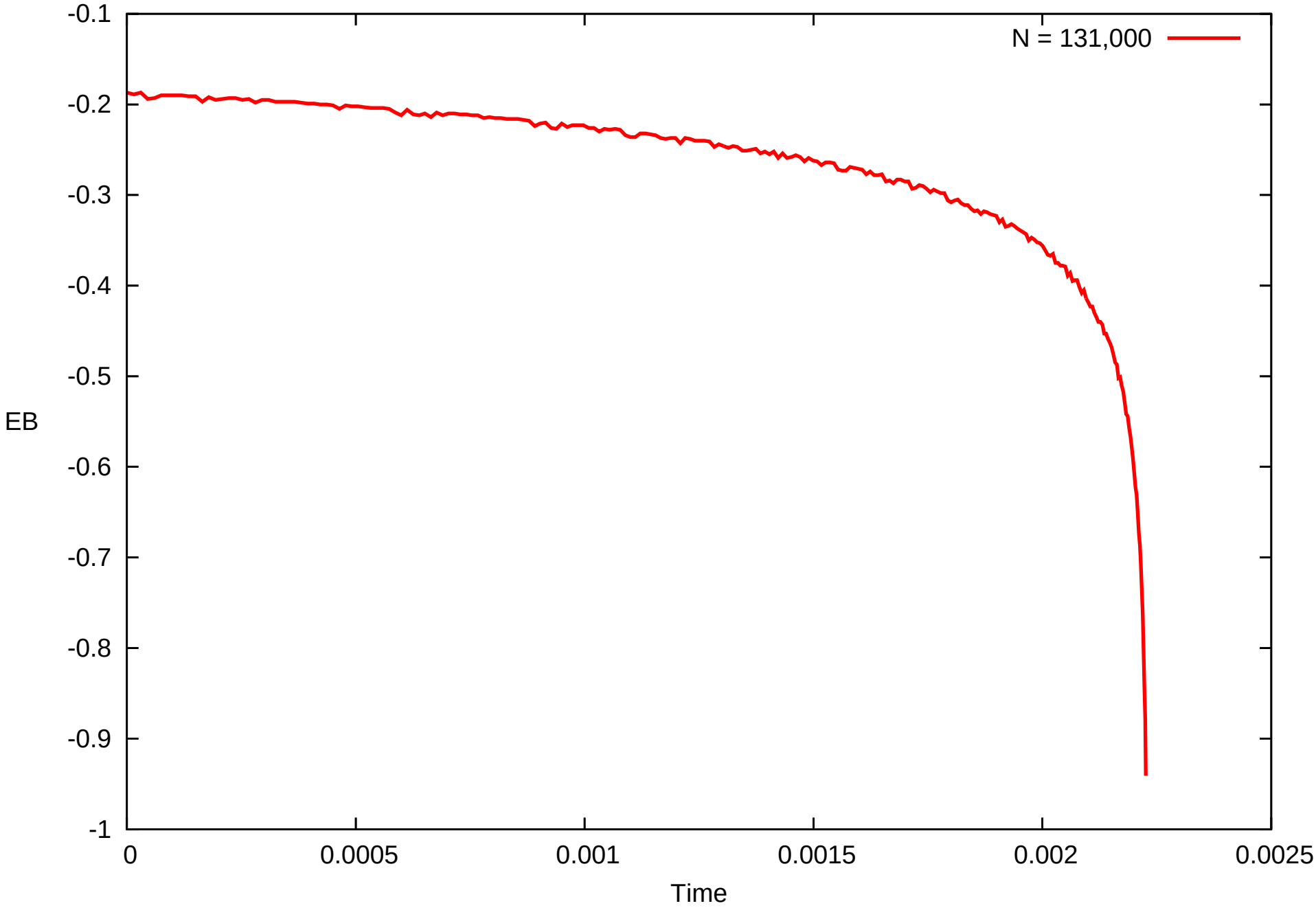
Graduated GR effect three stages: c^{-5}, c^{-2}, c^{-4}

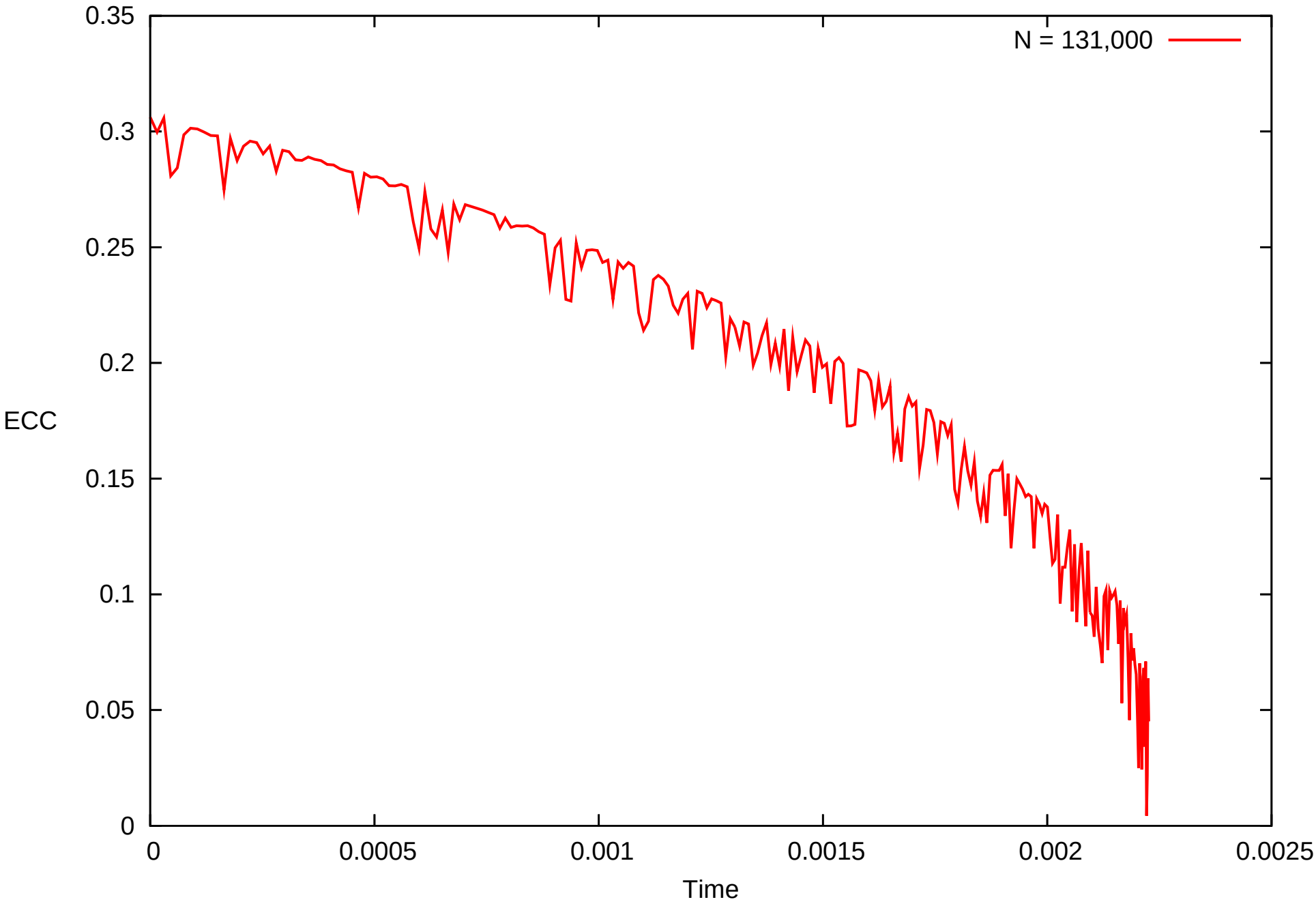
$$t_{\rm GR} \le 10\, t, \;\; t, \;\; 0.1\, t$$

Coalescence $$R < \frac{6\, M}{c^2}$$

Energy check $$E_{\rm tot} - \int \mathbf{P}_{\rm GR}\cdot\mathbf{v}\, dt = \text{const}$$


N = 131,000
-0.0005
-0.001
-0.0015
-0.002
-0.0025
-0.003
-0.0035
-0.004
-0.0045
EB
74
76
78
80
82
84
86
88
Time


N = 131,000
EB
Time
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1
0
0.0005
0.001
0.0015
0.002
0.0025


N = 131,000
ECC
Time
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.0005
0.001
0.0015
0.002
0.0025