Levi-Civita formulation

2D system: u_1, u_2

$$\begin{aligned} R_1 &= u_1^2 - u_2^2 \\ R_2 &= 2u_1u_2 \end{aligned}$$

Transformation

$$\mathbf{R} = \mathcal{L}(\mathbf{u})\mathbf{u}$$

Levi-Civita [1920] matrix

$$\mathcal{L}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{bmatrix} \qquad \Rightarrow \; R = u_1^2 + u_2^2$$

Definition $\dot{\mathbf{R}}^2 = \dot{\mathbf{R}}^T \cdot \dot{\mathbf{R}}$ with $\mathbf{R}' = 2\mathcal{L}(\mathbf{u})\mathbf{u}'$ and $\dot{R} = R'/R$

$$\dot{\mathbf{R}} = 2\mathcal{L}(\mathbf{u})\mathbf{u}'/R$$

$\dot{\mathbf{R}}^T = 2\mathbf{u}'\mathcal{L}^T(\mathbf{u})/R$ and $\mathcal{L}^T(\mathbf{u})\mathcal{L}(\mathbf{u}) = R\mathbf{I}$ give

$$\dot{\mathbf{R}}^T \cdot \dot{\mathbf{R}} = 4\mathbf{u}' \cdot \mathbf{u}'/R$$

Final equation of motion, with $\mathbf{u} \cdot \mathbf{u} = R$

$$\mathbf{u}'' = \frac{1}{2}h\mathbf{u} + \frac{1}{2}R\mathcal{L}^T(\mathbf{u})\mathbf{F}_{kl}$$

Binding energy per unit reduced mass

$$h = [(2\mathbf{u}' \cdot \mathbf{u}' - (m_k + m_l)]/R$$

Rate of change from $\dot{\mathbf{R}} \cdot \ddot{\mathbf{R}}$

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\mathbf{R}}^2 - \frac{(m_k + m_l)}{R}\right] = \dot{\mathbf{R}} \cdot \mathbf{F}_{kl}$$

Conversion by $h' = \mathbf{R}' \cdot \mathbf{F}_{kl}$ and $\dot{\mathbf{R}}$

$$h' = 2\mathbf{u}' \cdot \mathcal{L}^T(\mathbf{u})\mathbf{F}_{kl}$$

KS formulation

2D system: u_1, u_2

$$\begin{aligned} R_1 &= u_1^2 - u_2^2 \\ R_2 &= 2u_1u_2 \end{aligned}$$

Transformation

$$\mathbf{R} = \mathcal{L}(\mathbf{u})\mathbf{u}$$

Levi-Civita [1920] matrix

$$\mathcal{L}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{bmatrix} \qquad \Rightarrow \; R = u_1^2 + u_2^2$$

Matrix properties (Stiefel & Scheifele 1970)

$$\begin{aligned} \mathcal{L}^T(\mathbf{u})\mathcal{L}(\mathbf{u}) &= R\mathbf{I} \\ \mathcal{L}'(\mathbf{u}) &= \mathcal{L}(\mathbf{u}') \\ \mathcal{L}(\mathbf{u})\mathbf{v} &= \mathcal{L}(\mathbf{v})\mathbf{u} \\ \mathbf{u}\cdot\mathbf{u}\mathcal{L}(\mathbf{v})\mathbf{v} &- 2\mathbf{u}\cdot\mathbf{v}\mathcal{L}(\mathbf{u})\mathbf{v} + \mathbf{v}\cdot\mathbf{v}\mathcal{L}(\mathbf{u})\mathbf{u} = 0 \end{aligned}$$

Second & third properties give

$$\mathbf{R}' = 2\mathcal{L}(\mathbf{u})\mathbf{u}'$$

From $\mathcal{L}'(\mathbf{u}) = \mathcal{L}(\mathbf{u}')$

$$\mathbf{R}'' = 2\mathcal{L}(\mathbf{u})\mathbf{u}'' + 2\mathcal{L}(\mathbf{u}')\mathbf{u}'$$

Substituting $\mathbf{R}, \mathbf{R}'$, $R' = 2\mathbf{u}'\cdot\mathbf{u}$ and $n = 1$ in smoothed $\mathbf{R}''$

$$\begin{aligned} 2\mathbf{u}\cdot\mathbf{u}\mathcal{L}(\mathbf{u})\mathbf{u}'' &+ 2\mathbf{u}\cdot\mathbf{u}\mathcal{L}(\mathbf{u}')\mathbf{u}' - 4\mathbf{u}\cdot\mathbf{u}'\mathcal{L}(\mathbf{u})\mathbf{u}' + \\ &+ (m_k + m_l)\mathcal{L}(\mathbf{u})\mathbf{u} = (\mathbf{u}\cdot\mathbf{u})^3\mathbf{F}_{kl} \end{aligned}$$

Simplification by fourth property

$$2\mathbf{u}\cdot\mathbf{u}\mathcal{L}(\mathbf{u})\mathbf{u}''-2\mathbf{u}'\cdot\mathbf{u}'\mathcal{L}(\mathbf{u})\mathbf{u}+(m_k+m_l)\mathcal{L}(\mathbf{u})\mathbf{u}=(\mathbf{u}\cdot\mathbf{u})^3\mathbf{F}_{kl}$$

Multiply by $\mathcal{L}^{-1}(\mathbf{u})$ and first property

$$\mathbf{u}''+\{[(m_k+m_l)-2\mathbf{u}'\cdot\mathbf{u}']/2\mathbf{u}\cdot\mathbf{u}\}\mathbf{u}=\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\mathcal{L}^T(\mathbf{u})\mathbf{F}_{kl}$$

Definition $\dot{\mathbf{R}}^2=\dot{\mathbf{R}}^T\cdot\dot{\mathbf{R}}$ with $\mathbf{R}'$ and $\dot{R}=R'/R$

$$\dot{\mathbf{R}}=2\mathcal{L}(\mathbf{u})\mathbf{u}'/R$$

$\dot{\mathbf{R}}^T=2\mathbf{u}'\mathcal{L}^T(\mathbf{u})/R$ and orthogonality condition

$$\dot{\mathbf{R}}^T\cdot\dot{\mathbf{R}}=4\mathbf{u}'\cdot\mathbf{u}'/R$$

Final equation of motion, with $\mathbf{u}\cdot\mathbf{u}=R$

$$\mathbf{u}''=\frac{1}{2}h\mathbf{u}+\frac{1}{2}R\mathcal{L}^T(\mathbf{u})\mathbf{F}_{kl}$$

Binding energy per unit reduced mass

$$h=[(2\mathbf{u}'\cdot\mathbf{u}'-(m_k+m_l)]/R$$

Rate of change from $\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}}$

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\mathbf{R}}^2-\frac{(m_k+m_l)}{R}\right]=\dot{\mathbf{R}}\cdot\mathbf{F}_{kl}$$

Conversion by $h'=\mathbf{R}'\cdot\mathbf{F}_{kl}$ and $\dot{\mathbf{R}}$

$$h'=2\mathbf{u}'\cdot\mathcal{L}^T(\mathbf{u})\mathbf{F}_{kl}$$

Generalized 4×4 matrix

$$\mathcal{L}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix}$$

Explicit components of $\mathbf{R}$

$$\begin{aligned} R_1 &= u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ R_2 &= 2(u_1 u_2 - u_3 u_4) \\ R_3 &= 2(u_1 u_3 + u_2 u_4) \\ R_4 &= 0 \end{aligned}$$

Summing the squares and square root

$$R = u_1^2 + u_2^2 + u_3^2 + u_4^2$$

Case $R_1 > 0$: combine R_1 and R

$$u_1^2 + u_4^2 = \frac{1}{2}(R_1 + R)$$

Redundancy $u_4 = 0$

$$\begin{aligned} u_1 &= [\frac{1}{2}(R_1 + R)]^{1/2} \\ u_2 &= \frac{1}{2} R_2 / u_1 \\ u_3 &= \frac{1}{2} R_3 / u_1 \end{aligned}$$

Case $R_1 < 0$: subtract R_1 from R

$$u_2^2 + u_3^2 = \frac{1}{2}(R - R_1)$$

Redundancy $u_3 = 0$

$$\begin{aligned} u_2 &= [\frac{1}{2}(R - R_1)]^{1/2} \\ u_1 &= \frac{1}{2}R_2/u_2 \\ u_4 &= \frac{1}{2}R_3/u_2 \end{aligned}$$

Regularized velocity: invert $\mathbf{R}'$ and use first property

$$\mathbf{u}' = \frac{1}{2}\mathcal{L}^T(\mathbf{u})\mathbf{R}'/R = \frac{1}{2}\mathcal{L}^T(\mathbf{u})\dot{\mathbf{R}}$$

Final equations of motion

$$\begin{aligned} \mathbf{u}'' &= \frac{1}{2}h\,\mathbf{u} + \frac{1}{2}R\,\mathcal{L}^T\,\mathbf{F}_{kl} \\ h' &= 2\,\mathbf{u}' \cdot \mathcal{L}^T\,\mathbf{F}_{kl} \\ t' &= \mathbf{u} \cdot \mathbf{u} \end{aligned}$$

Semi-major axis

$$a = -\frac{1}{2}(m_k + m_l)/h$$

Eccentricity: $R = a(1 - e\cos\theta)$ and $nt = \theta - e\sin\theta$

$$e^2 = (1 - R/a)^2 + 4(\mathbf{u} \cdot \mathbf{u}')^2/[(m_k + m_l)a]$$

Bilinear relation: $\dot{R}_4 = 0$

$$u_4u_1' - u_3u_2' + u_2u_3' - u_1u_4' = 0$$

Unperturbed two-body motion

Maximum force:

$$j = \max_i \left(m_i/|\mathbf{r}_i - \mathbf{r}_{cm}|^2\right), \qquad i = 1, N$$

Smallest inverse travel time

$$\beta_s = \mathbf{r}_s \cdot \dot{\mathbf{r}}_s / r_s^2, \qquad \mathbf{r}_s - \mathbf{r}_{cm} \Rightarrow \mathbf{r}_s$$

Perturber boundary

$$r_\gamma = R[2\tilde{m}/(m_b \gamma_{\min})]^{1/3}$$

Travel time: $\dot{r}_s < 0$

$$\Delta t_{\mathrm{in}} = (r_s - r_\gamma)/|\dot{r}_s|\,,$$

Free-fall time

$$\Delta t_a = [2\Delta t_{\mathrm{in}} \dot{r}_s r_s^2/(m_b + m_s)]^{1/2}$$

Return time of dominant body

$$\Delta t_j = [2(r_j - r_\gamma) r_j^2/(m_b + m_j)]^{1/2}$$

Unperturbed time interval

$$\Delta t_\gamma = \min\left(\Delta t_{\mathrm{in}}, \Delta t_a, \Delta t_j, \Delta t_{cm}\right)$$

Unperturbed periods

$$K = 1 + \frac{1}{2}\Delta t_\gamma / t_K$$

Final time interval

$$\Delta t = K \min\left(t_K, \Delta t_{\mathrm{cm}}\right)$$

KS algorithms

Perturber prediction

$$\begin{aligned}\mathbf{r}_j &= ((\frac{1}{6}\mathbf{F}^{(1)}\,\delta t'_j + \frac{1}{2}\mathbf{F})\,\delta t'_j + \mathbf{v}_0)\,\delta t'_j + \mathbf{r}_0 \\ \dot{\mathbf{r}}_j &= ((\frac{1}{2}\mathbf{F}^{(1)}\,\delta t'_j + \mathbf{F})\,\delta t'_j + \mathbf{v}_0\,, \quad \delta t'_j = t - t_j\end{aligned}$$

KS prediction

$\mathbf{u}$ and $\mathbf{u}'$ to order $\mathbf{u}^{(5)}$

Basic Hermite

Stabilization factor in $\mathbf{u}''$

h predicted to order $h^{(2)}$

KS transformations

Global coordinates and velocities $\mathbf{r}_k, \mathbf{r}_l,\, \dot{\mathbf{r}}_k, \dot{\mathbf{r}}_l$

Physical perturbation

$\mathbf{P}$ and $\dot{\mathbf{P}}$ due to perturbers, set $\mathbf{P}' = R\,\dot{\mathbf{P}}$

$j > N$: c.m. approximation or components

Slow-down factor

Include κ in $\mathbf{P}$ and $\dot{\mathbf{P}}$, also $t' = \kappa\,\mathbf{u}\cdot\mathbf{u}$

Energy prediction (Stumpff method)

h to order $h^{(4)}$

KS corrector

$\mathbf{u}, \mathbf{u}'$ to order $\mathbf{u}^{(5)}$ and h to $h^{(4)}$

Time derivatives

Taylor series $t'' = 2\mathbf{u}\cdot\mathbf{u}', ..., t^{(6)} = 2\mathbf{u}\cdot\mathbf{u}^{(5)} + ...$

Hermite KS

Standard KS

$$
\begin{aligned}
\mathbf{u}'' &= \frac{1}{2} h\, \mathbf{u} + \frac{1}{2} R\, \mathcal{L}^T\, \mathbf{F}_{kl} \\
h' &= 2\, \mathbf{u}' \cdot \mathcal{L}^T\, \mathbf{F}_{kl} \\
t' &= \mathbf{u} \cdot \mathbf{u}
\end{aligned}
$$

New notation

$$
\begin{aligned}
\mathbf{F}_{\mathrm{u}} &= \mathbf{u}'' \\
\mathbf{Q} &= \mathcal{L}^T\, \mathbf{P}\,,
\end{aligned}
$$

with $\mathbf{P} = \mathbf{F}_{kl}$ as the perturbing force.

Basic equations

$$
\begin{aligned}
\mathbf{F}_{\mathrm{u}} &= \frac{1}{2} h\, \mathbf{u} + \frac{1}{2} R\, \mathbf{Q} \\
h' &= 2\, \mathbf{u}' \cdot \mathbf{Q} \\
t' &= \mathbf{u} \cdot \mathbf{u}
\end{aligned}
$$

Hermite $\mathbf{F}$, $\mathbf{F}'$ formulation

$$
\begin{aligned}
\mathbf{F}_{\mathrm{u}} &= \frac{1}{2} h\, \mathbf{u} + \frac{1}{2} R\, \mathbf{Q} \\
\mathbf{F}'_{\mathrm{u}} &= \frac{1}{2} (h' \mathbf{u} + h \mathbf{u}' + R' \mathbf{Q} + R \mathbf{Q}') \\
h' &= 2\, \mathbf{u}' \cdot \mathbf{Q} \\
h'' &= 2 \mathbf{F}_{\mathrm{u}} \cdot \mathbf{Q} + 2 \mathbf{u}' \cdot \mathbf{Q}' \\
t' &= \mathbf{u} \cdot \mathbf{u}
\end{aligned}
$$

The derivatives of $\mathbf{P}$, $\mathbf{Q}$ and t' are readily available. Note that $\mathbf{P}' = R\dot{\mathbf{P}}$ and that $\mathcal{L}^T(\mathbf{u}')$ can be obtained by substituting $\mathbf{u}'$ for $\mathbf{u}$. For implementation, significant accuracy can be gained by high-order prediction (not used in standard Hermite).

N-body interface

Centre of mass acceleration

$$\ddot{\mathbf{r}}_{cm} = (m_k \mathbf{F}_k + m_l \mathbf{F}_l)/(m_k + m_l)$$

Global coordinates

$$\begin{aligned} \mathbf{r}_k &= \mathbf{r}_{cm} + \mu \mathbf{R}/m_k \\ \mathbf{r}_l &= \mathbf{r}_{cm} - \mu \mathbf{R}/m_l \end{aligned}$$

Relative perturbation

$$\gamma = |\mathbf{F}_k - \mathbf{F}_l| R^2/(m_k + m_l)$$

Tidal approximation

$$r_\gamma = R[2\tilde{m}/(m_k + m_l)\gamma_{\rm min}]^{1/3}, \qquad \gamma_{min} \simeq 10^{-6}$$

Perturber selection

$$r_{ij} < r_\gamma, \qquad R = a(1+e)$$

Regularized time-step

$$\Delta\tau = \eta_u (1/2|h|)^{1/2} 1/(1 + 1000\gamma)^{1/3}$$

Physical time-step

$$\Delta t = \sum_{k=1}^{n} \frac{1}{k!} t_0^{(k)} \Delta\tau^k, \qquad n = 6$$

Time derivatives

$$\begin{aligned} t_0'' &= 2\mathbf{u}' \cdot \mathbf{u} \\ t_0^{(3)} &= 2\mathbf{u}'' \cdot \mathbf{u} + 2\mathbf{u}' \cdot \mathbf{u} \end{aligned}$$